

# Complex 2D Matrix Model and Geometrical Map on Complex- $N_c$ Plane

Kanabu Nawa<sup>1</sup>, Sho Ozaki<sup>1</sup>, Hideko Nagahiro<sup>2,3</sup>, Daisuke Jido<sup>4,5</sup> and Atsushi Hosaka<sup>3</sup>

<sup>1</sup>Quantum Hadron Physics Laboratory, RIKEN Nishina Center, Saitama 351-0198, Japan

<sup>2</sup>Department of Physics, Nara Women's University, Nara 630-8506, Japan

<sup>3</sup>Research Center for Nuclear Physics (RCNP), Osaka University, Osaka 567-0047, Japan

<sup>4</sup>Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

<sup>5</sup>J-PARC Branch, KEK Theory Center, Institute of Particle and Nuclear Studies, Ibaraki 319-1106, Japan

We study the parameter dependence of the internal structure of resonance states by formulating Complex two-dimensional (2D) Matrix Model, where the two dimensions represent two-levels of resonances. We calculate a critical value of the parameter at which “nature transition” with character exchange occurs between two resonance states, from the viewpoint of geometry on complex-parameter space. Such critical value is useful to know the internal structure of resonance states with variation of the parameter in the system. We apply the model to analyze the internal structure of hadrons with variation of the color number  $N_c$  from  $\infty$  to a realistic value 3. By regarding  $1/N_c$  as the variable parameter in our model, we calculate a critical color number of nature transition between hadronic states in terms of quark-antiquark pair and mesonic molecule as exotics from the geometry on complex- $N_c$  plane. For the large- $N_c$  effective theory, we employ the chiral Lagrangian induced by holographic QCD with D4/D8/ $\overline{\text{D8}}$  multi-D brane system in the type IIA superstring theory.

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How do characters of states change with variation of a parameter which specifies the property of the system or of the environment where the system is placed? This is a central issue discussed in various phenomena of physics, e.g., deformed nuclei depending on the deformation parameter of nuclear mean-field potential [1], electronic wave function configurations of diatomic molecules depending on the internuclear distance [2], and conversion of solar neutrinos depending on the distance from the center of the sun [3]. In quantum mechanics, one may start with a hermite model Hamiltonian  $\hat{\mathcal{H}}(\lambda)$  with a real parameter  $\lambda$ . Here one can assume that the eigenstates of  $\hat{\mathcal{H}}$  at  $\lambda = 0$ ,  $\phi_i$  ( $i = 1, 2, \dots$ ), can be an appropriate basis to classify the properties of the eigenstates for finite  $\lambda$ ,  $\psi_i(\lambda)$  ( $i = 1, 2, \dots$ ), in terms of the characters of  $\phi_i$ . Now, if the energy expectation values  $\varepsilon_i(\lambda) \equiv \langle \phi_i | \hat{\mathcal{H}}(\lambda) | \phi_i \rangle$  ( $i = 1, 2, \dots$ ) cross with each other at a certain value  $\lambda = \lambda_t \in \mathbf{R}$ , the energy eigenvalues  $E_i(\lambda)$  of  $\psi_i(\lambda)$  have level repulsion, i.e., anticrossing at  $\lambda_t$  (see Fig. 1(a)) due to the Neumann-Wigner non-crossing rule [4]. At this point, the overlap between  $\psi_i(\lambda)$  and  $\phi_i$  is exceeded by that between  $\psi_i(\lambda)$  and  $\phi_j$  as  $|\langle \phi_i | \psi_i \rangle|^2 \leq |\langle \phi_j | \psi_i \rangle|^2$ . Therefore, due to orthogonality,  $\psi_i(\lambda)$  and  $\psi_j(\lambda)$  exchange their characters in terms of the appropriate basis  $\phi_i$  and  $\phi_j$  at the anticrossing point  $\lambda = \lambda_t$ , which we call “nature transition” in this paper. In fact, the critical value  $\lambda_t$  is very important to know or classify the internal structure of the quantum states which can change with variation of certain parameter  $\lambda$ .

In this paper, we consider the parameter dependence of states in quantum systems with dissipation into decay channels outside of the model space. Such systems are often called open quantum systems with resonance states, which are effectively described by non-hermite model Hamiltonians  $\hat{\mathcal{H}}(\lambda)$  with complex energy eigenvalues [5]. The real and imaginary parts of the energy correspond to the mass and decay width of the resonance states, respectively. In such open quantum

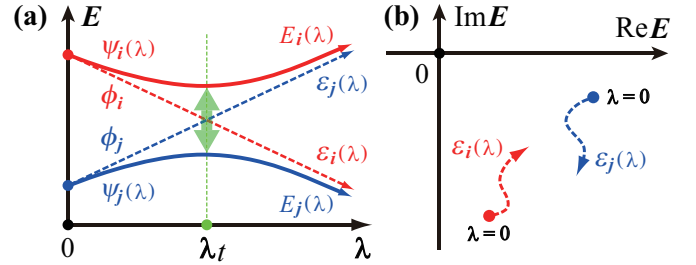


FIG. 1: (Color) (a) Anticrossing between  $i$ th and  $j$ th eigenstates of hermite Hamiltonian  $\hat{\mathcal{H}}(\lambda)$  with variation of  $\lambda \in \mathbf{R}$ . Indices of lines are explained in the text.  $i$ th and  $j$ th eigenstates exchange their characters at anticrossing point  $\lambda = \lambda_t$  as nature transition. (b)  $\varepsilon_i(\lambda)$  and  $\varepsilon_j(\lambda)$  of non-hermite Hamiltonian  $\hat{\mathcal{H}}(\lambda)$  with variation of  $\lambda \in \mathbf{R}$  on complex energy plane.  $\varepsilon_i(\lambda)$  and  $\varepsilon_j(\lambda)$  generally have no degeneracy at certain value of  $\lambda \in \mathbf{R}$  except for accidental case.

systems,  $\varepsilon_i(\lambda)$  of  $\phi_i$  can move on the complex energy plane (see Fig. 1(b)) without having degeneracy at a certain value of  $\lambda$  except for an accidental case [6]. Therefore, a simple criterion should be newly found to judge the existence of the nature transition between the resonances, and its critical value  $\lambda_t$  can be used to know or classify the internal structure of the resonance states depending on the parameter. In this paper, we construct Complex two-dimensional (2D) Matrix Model to discuss the nature transition between two resonance states. (Two dimensions represent two levels of resonances.) This 2D model will give an elementary understanding for higher dimensional problems because the latter can often be reduced locally to the 2D problems. We show that, by extending  $\lambda$  to a *complex variable*, the geometry on the complex- $\lambda$  plane can give the criterion of the nature transition within the real parameter subspace  $\lambda \in \mathbf{R}$ .

After establishing the general framework, we apply it to the hadron physics with strong interaction, which is governed

by *quantum chromodynamics* (QCD) as the  $SU(N_c)$  gauge theory with color number  $N_c = 3$  [7]. By extending  $N_c$  to an arbitrary number,  $1/N_c$ -expansion provides a systematic perturbative treatment. The leading order as “large- $N_c$  QCD” reproduces lots of QCD phenomenologies [8, 9]. In fact, in large- $N_c$  QCD, the internal structure of mesons becomes clear: mesons as quark-antiquark ( $q\bar{q}$ ) pairs appear with masses of  $O(N_c^0)$  and zero widths, while “mesonic molecules” can also appear as resonances with masses and widths increasing along with  $N_c$  because the meson-meson interactions are suppressed with  $O(N_c^{-1})$  [10]. From such considerations in large- $N_c$ , one may say that exotics can be suppressed in the real world [9]. However the internal structures of hadrons should depend on  $N_c$ , and, by decreasing  $N_c$  from  $\infty$  to the realistic value 3, the states can easily exchange their characters via nature transitions. Here, an essential question arises: *what is the internal structure of each state with variation of  $N_c$  from  $\infty$  to 3?* By regarding  $q\bar{q}$  and mesonic molecule states in large- $N_c$  as the appropriate basis  $\phi_i$  ( $i = 1, 2$ ), and by identifying  $1/N_c$  to  $\lambda$  in the Complex 2D Matrix Model, we will calculate a critical color number of nature transitions in terms of appropriate basis from the geometry on the complex- $N_c$  plane. As an example, we investigate the internal structure of  $a_1(1260)$  meson with admixed nature of  $q\bar{q}$  and  $\pi\rho$ -molecule components. For the large- $N_c$  effective theory, we employ the chiral Lagrangian induced by holographic QCD with D4/D8/ $\overline{D8}$  multi-D brane system in the type IIA superstring theory [11, 12].

First we formulate Complex 2D Matrix Model to treat a two-level problem in a quantum system with resonances. We describe resonance states by using the bi-orthogonal representation as  $|\phi_i\rangle$  ( $i = 1, 2$ ): its bra-state is defined by the complex conjugate of the Dirac bra-state  $\langle\phi_i| \equiv \langle\phi_i^*|$ , which is firstly introduced to describe the unstable nuclei in nuclear physics [13–15]. Only by taking such bi-orthogonal representation, resonance states with different eigenvalues are orthogonal to each other as  $\langle\phi_i|\phi_j\rangle = \delta_{ij}$ , which is needed to employ the matrix representation of operators in such basis. As anticipated, we suppose that  $|\phi_i\rangle$ , the eigenstates of  $\hat{\mathcal{H}}(0)$  at  $\lambda = 0$ , are the appropriate basis with clear characters and are useful to classify the quantum states. Hence we consider the Hamilton matrix  $\mathcal{H}(\lambda) \equiv [(\phi_i|\hat{\mathcal{H}}(\lambda)|\phi_j)]$  in this basis:

$$\mathcal{H}(\lambda) = \begin{pmatrix} \varepsilon_1(\lambda) & V_{12}(\lambda) \\ V_{21}(\lambda) & \varepsilon_2(\lambda) \end{pmatrix}, \quad (1)$$

where  $\varepsilon_i \in \mathbf{C}$  is the energy of  $|\phi_i\rangle$  and  $V_{ij} \in \mathbf{C}$  are the interaction satisfying  $V_{ij}(0) = 0$ .  $\lambda \in \mathbf{R}$  is a parameter, controlling the development of the two eigenstates  $|\psi_i(\lambda)\rangle$  which can be obtained in terms of the basis  $|\phi_i\rangle$  as

$$|\psi_i(\lambda)\rangle \equiv C_{i1}(\lambda)|\phi_1\rangle + C_{i2}(\lambda)|\phi_2\rangle. \quad (i = 1, 2) \quad (2)$$

The coefficients  $C_{ij}(\lambda)$  carry the information for the internal structure of the eigenstates  $|\psi_i(\lambda)\rangle$  in terms of  $|\phi_i\rangle$ . There is a subtlety for the interpretation of component weights from  $C_{ij}(\lambda)$ , since the norms  $\langle\psi_i|\psi_i\rangle = C_{i1}^2 + C_{i2}^2$  can be complex numbers due to the bi-orthogonality. Several attempts

have been made to interpret such complex probability of resonances (for example, see Ref. [16]), while a consensus has not been achieved yet. In this work we simply presume the module,  $|C_{ij}(\lambda)|^2$ , to be interpreted as the component weights, as it is suitable for narrow resonances. At  $\lambda = 0$ ,  $|\psi_i(\lambda)\rangle$  coincides with  $|\phi_i\rangle$  due to  $V_{ij}(0) = 0$ , so that  $C_{ij}(0) = 0$  for  $i \neq j$ .

Now, if  $\hat{\mathcal{H}}(\lambda)$  is hermite with real eigenvalues, the level crossing of  $|\phi_i\rangle$  is known to give the level anticrossing of  $|\psi_i(\lambda)\rangle$  as shown in Fig. 1(a) [4]. At this anticrossing point  $\lambda = \lambda_t$ ,  $|\psi_i(\lambda)\rangle$  exchange their characters as “nature transition” with the transition condition  $|C_{i1}(\lambda_t)|^2 = |C_{i2}(\lambda_t)|^2$ , where the two basis components  $|\phi_1\rangle$  and  $|\phi_2\rangle$  are equally mixed as a character exchanging point. In this paper, we newly consider the case that  $\mathcal{H}(\lambda)$  is non-hermite with complex eigenvalues for resonance states. As we will show below,  $|C_{i1}|^2 = |C_{i2}|^2$  can be satisfied *at least* by the energy coincidence  $E_1 = E_2$ , which can be realized if one extends  $\lambda$  to a *complex variable* [17]. Therefore, to get a geometrical insight for the existence of nature transition, here we introduce the *complex- $\lambda$  plane*.

By solving the Schrödinger equation:  $\hat{\mathcal{H}}|\psi\rangle = E|\psi\rangle$ , we find the eigenvalues  $E_i(\lambda)$  ( $i = 1, 2$ ) as

$$E_i(\lambda) = \{\varepsilon_1(\lambda) + \varepsilon_2(\lambda)\}/2 \pm F(\lambda), \quad (3)$$

$$F(\lambda) \equiv \sqrt{A(\lambda)^2 + \overline{V}(\lambda)^2}, \quad (4)$$

$$A(\lambda) \equiv \{\varepsilon_1(\lambda) - \varepsilon_2(\lambda)\}/2, \quad (5)$$

$$\overline{V}(\lambda)^2 \equiv V_{12}(\lambda)V_{21}(\lambda), \quad (6)$$

and the coefficient ratios  $R_i(\lambda)$  of the eigenstates  $|\psi_i(\lambda)\rangle$  ( $i = 1, 2$ ) in Eq. (2) as

$$R_i(\lambda) \equiv \frac{C_{i2}(\lambda)}{C_{i1}(\lambda)} = -\frac{1}{V_{12}(\lambda)}\{A(\lambda) \mp F(\lambda)\}. \quad (7)$$

The upper (lower) sign in Eqs. (3) and (7) corresponds to  $i = 1$  ( $i = 2$ ). The ratios (7) are sufficient to discuss the nature transition between two levels as below.

Now we consider the transition condition  $|C_{i1}(\lambda)|^2 = |C_{i2}(\lambda)|^2$  on the complex- $\lambda$  plane. Due to the bi-orthogonality  $\langle\psi_1|\psi_2\rangle = 0$ , i.e.,  $R_1 R_2 = -1$ , the transition condition can be written only by the ratios (7) as  $|R_1(\lambda)| = |R_2(\lambda)|$ , which is equivalent from Eq. (7) to

$$\text{Re}[A(\lambda)^* F(\lambda)] = 0. \quad (8)$$

Due to the square root in (4), Eq. (8) becomes equivalent to the two conditions:

$$\text{Re}[A(\lambda)^* \overline{V}(\lambda)] = 0, \quad (9)$$

$$|A(\lambda)|^4 - \{\text{Im}[A(\lambda)^* \overline{V}(\lambda)]\}^2 \leq 0. \quad (10)$$

From Eq. (9),  $|\text{Im}[A(\lambda)^* \overline{V}(\lambda)]| = |A(\lambda)||\overline{V}(\lambda)|$ , so that the condition (10) becomes

$$|A(\lambda)|^2 \leq |\overline{V}(\lambda)|^2, \quad (11)$$

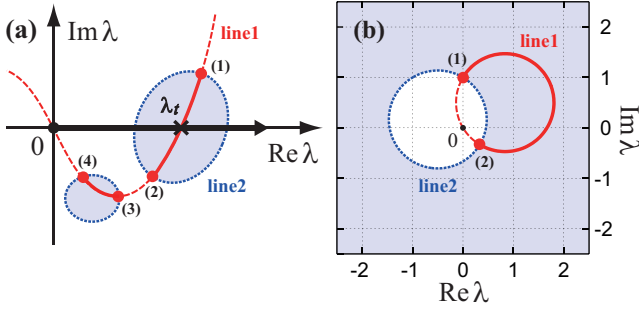


FIG. 2: (Color) (a) Schematic figure of geometrical map with transition lines and exceptional points on complex  $\lambda$  plane. Line 1 and shaded area with boundary of line 2 correspond to the conditions (9) and (11), respectively. Points (n) denote the exceptional points  $\lambda_{\text{EX}}^{(n)}$ . Transition lines are shown by the solid curves, which satisfy both (9) and (11). (b) Linear- $\lambda$  model with  $\varepsilon_1(0) = 1000 - 200i$ ,  $\varepsilon_2(0) = 1200$ ,  $v_{11} = 0$ ,  $v_{22} = -100 - 200i$ ,  $v_{12} = v_{21} = 200 + 50i$  in MeV unit as test values.

which has been divided by  $|A(\lambda)|^2$  since  $\lambda$ 's for  $A(\lambda) = 0$  trivially satisfy the conditions (9) and (11). Now the “transition line” is defined as the region satisfying  $|C_{i1}(\lambda)|^2 = |C_{i2}(\lambda)|^2$ , i.e., both conditions (9) and (11) on the complex- $\lambda$  plane. Therefore, the line (9), named “line 1”, can be the candidate of the transition line, and the region (11) with the boundary  $|A(\lambda)|^2 = |\overline{V}(\lambda)|^2$ , named “line 2”, selects the proper area for the transition line. The region (11) always excludes the origin  $\lambda = 0$  for the case  $\varepsilon_1(0) \neq \varepsilon_2(0)$ , because  $|A(0)| > 0$  and  $|\overline{V}(0)| = 0$ . Then, if the transition line crosses the real- $\lambda$  axis, the nature transition occurs at the crossing point  $\lambda = \lambda_t \in \mathbf{R}$  (see schematic Fig. 2(a)).

Now, from (9) and (11), the crossing points  $\lambda = \lambda_{\text{EX}}^{(n)} \in \mathbf{C}$  ( $n = 1, 2, \dots$ ) of line 1 and line 2 satisfy the condition  $A(\lambda)^* F(\lambda) = 0$  for  $\forall A(\lambda) \neq 0$ , which is equivalent to

$$F(\lambda)^2 = A(\lambda)^2 + \overline{V}(\lambda)^2 = 0. \quad (12)$$

Therefore, at  $\lambda = \lambda_{\text{EX}}^{(n)}$ , the mass difference in Eq. (3) becomes zero and two eigenvalues coincide as  $E_1(\lambda) = E_2(\lambda)$ .  $\lambda_{\text{EX}}^{(n)}$  are called the “exceptional points” on the complex- $\lambda$  plane [17]. In fact, the importance of the exceptional points has been intensively studied both theoretically [18] and experimentally [19] in quantum chaos, where the dense exceptional points on the complex- $\lambda$  plane correspond to the development of quantum chaos in the energy-level statistics [18]. Now, in this paper, we can show that line 1 and line 2 cross each other at all exceptional points, so that these points can be the *end points* of the transition lines. Therefore, the location of the exceptional points is very important to geometrically judge the existence of  $\lambda_t \in \mathbf{R}$ .

One simple example is the “linear- $\lambda$  model” with  $\varepsilon_i(\lambda) = \varepsilon_i(0) + \lambda v_{ii}$  and  $V_{ij}(\lambda) = \lambda v_{ij}$  ( $i, j = 1, 2$  and  $i \neq j$ ). Two exceptional points and one transition line appear (see Fig. 2(b)), which are checked from the power counting about  $\lambda$  in (12). In this model, Eq. (9) can be equally written with

$$\Delta\varepsilon \equiv \varepsilon_1(0) - \varepsilon_2(0), \Delta v \equiv v_{11} - v_{22}, \text{ and } \overline{v}^2 \equiv v_{12}v_{21} \text{ as}$$

$$|\lambda - \overline{\lambda}| = |\overline{\lambda}|, \quad (\overline{\lambda} \equiv -(\Delta\varepsilon)\overline{v}^* \{2\text{Re}[(\Delta v)\overline{v}^*]\}^{-1}) \quad (13)$$

so that line 1 is a circle crossing at  $\lambda = 0$ ,  $\lambda_{\text{EX}}^{(1)}$  and  $\lambda_{\text{EX}}^{(2)}$ , and the transition line has an arc shape. Ref. [18] shows that, in the linear- $\lambda$  model, the eigenvalue behaviors for  $\lambda \in \mathbf{R}$  depend on the location of the two exceptional points; if the two locate on the opposite sides striding over the real- $\lambda$  axis, level anticrossing/width crossing occurs, while, if not, level crossing/width crossing occurs. Therefore, in Fig. 2(b), we can newly show that the nature transition occurs only in the level anticrossing/width crossing case. In this way, as for the linear- $\lambda$  model with two exceptional points, we can relate the behaviors of poles on the complex-energy plane and their internal structures through the geometry on the complex- $\lambda$  plane. The linear- $\lambda$  model also suggests that, if  $v_{11} = v_{22} = 0$ , the radius of the circle of line 1 in Eq. (13) diverges:  $|\overline{\lambda}| \rightarrow \infty$ , so that there is no nature transition for finite  $\lambda$ . There only occurs the mixing of the basis components up to 50% at most. Therefore the  $\lambda$ -dependence in the diagonal components of the matrix form (1) is needed to have the nature transition between resonance states.

So far, we have formulated the model described by the Hamiltonian (1) with arbitrary complex functions:  $\varepsilon_i(\lambda)$  and  $V_{ij}(\lambda)$ , to treat the general two-level problems on the complex-energy plane. We have shown that the geometrical map on the complex- $\lambda$  plane provides the geometrical insight for the existence of nature transition within the real parameter subspace  $\lambda \in \mathbf{R}$ .

Let us now apply the Complex 2D Matrix Model to the hadron physics. We take, as an example, the  $a_1(1260)$  meson which has admixed nature of  $q\bar{q}$  and  $\pi\rho$ -molecule components. First, we prepare the appropriate basis for the  $q\bar{q}$  and the  $\pi\rho$ -molecule states in large- $N_c$ . For the large- $N_c$  effective theory, we make use of the chiral Lagrangian induced by holographic QCD with D4/D8/ $\overline{\text{D8}}$  multi-D brane system in the type IIA superstring theory [11, 12]. Due to the large- $N_c$  condition of the duality with “classical” supergravity, the  $a_1$  meson appearing as a gauge field in holographic QCD should correspond to the  $q\bar{q}$  state. On the other hand, the holographic action also induces the energy-dependent  $\pi\rho$  interaction as the Weinberg-Tomozawa (WT) interaction of order  $O(N_c^{-1})$ . Due to its attractive interaction, the non-perturbative  $\pi\rho$  dynamics gives a resonance pole as the “ $\pi\rho$ -molecule state”. The  $a_1$  meson as the  $\pi\rho$ -molecule is also studied in the chiral unitary model [20, 21]. Thus, by preparing the  $q\bar{q}$  and  $\pi\rho$ -molecule states as the appropriate basis  $\phi_i$  ( $i = 1, 2$ ) and identifying  $1/N_c$  to  $\lambda$  in the Complex 2D Matrix Model, we will calculate the critical color number of the nature transition from the geometry on the complex- $N_c$  plane. Below, we investigate the scattering equation for the  $\pi\rho$  propagator in the  $J^P = 1^+$  channel. By reducing the relativistic eigenvalue equation to the Schrödinger equation of the model (1) with a non-relativistic approximation as below, we will derive the geometrical map on the complex- $N_c$  plane for the  $a_1$  meson.



FIG. 3: Interactions between  $\pi$ ,  $\rho$  and  $a_1$  mesons; (a) three-point interaction and (b) Weinberg-Tomozawa interaction.

From the Lagrangian in holographic QCD [11, 12], we obtain the three-point interaction  $v_{a_1\pi\rho}$  and the WT interaction  $v_{WT}$  in Fig. 3 after proper  $s$ -wave projection [22] in the form,

$$v_{a_1\pi\rho} = 2\sqrt{2}f_\pi^{-1}g_{a_1\pi\rho}(s - m_\rho^2), \quad (14)$$

$$v_{WT} = -4^{-1}f_\pi^{-2}\{3s - 2(m_\rho^2 + m_\pi^2) - (m_\rho^2 - m_\pi^2)^2 s^{-1}\}. \quad (15)$$

By taking the two experimental inputs, e.g.,  $f_\pi = 92.4\text{MeV}$  and  $m_\rho = 776\text{MeV}$ , all the masses and coupling constants of hadrons can be uniquely determined in the holographic approach as  $m_{a_1} = 1189\text{MeV}$  and  $g_{a_1\pi\rho} = 0.26$ . (In the D4/D8/ $\overline{\text{D8}}$  model, pion is exactly massless, whereas we use an isospin-averaged mass value:  $m_\pi = 138\text{MeV}$ .)

Now we introduce a two-dimensional  $G$ -function with  $\pi\rho$  and  $q\bar{q}$  channels, having  $J^P = 1^+$  as the  $a_1$  meson:

$$G^{-1} = G_0^{-1} - V \quad (16)$$

$$= \begin{pmatrix} G_{\pi\rho} & 0 \\ 0 & G_{a_1} \end{pmatrix}^{-1} - \begin{pmatrix} v_{WT} & v_{a_1\pi\rho} \\ v_{a_1\pi\rho} & 0 \end{pmatrix}, \quad (17)$$

where  $G_{a_1} \equiv (s - m_{a_1}^2)^{-1}$  is a propagator for the  $q\bar{q}$  state as the  $a_1$  meson and  $G_{\pi\rho}$  is  $\pi\rho$  loop function [20] as

$$G_{\pi\rho} \equiv \int \frac{d^4q}{(2\pi)^4} \frac{1}{(P-q)^2 - m_\pi^2 + i\epsilon} \frac{1}{q^2 - m_\rho^2 + i\epsilon}, \quad (18)$$

with  $P$  a total incident momentum as  $P^2 = s$ . We use a dimensional regularization with the natural condition [23] to avoid the effect of CDD pole in Eq. (18). In fact, the loop integral of Eq. (18) appears in the scattering equation of the T-matrix with the separable approximation for the interactions [20]. Then one can sum up the diagonal component of the potential in Eq. (17) as

$$G^{-1} = \begin{pmatrix} G_{WT} & 0 \\ 0 & G_{a_1} \end{pmatrix}^{-1} - \begin{pmatrix} 0 & v_{a_1\pi\rho} \\ v_{a_1\pi\rho} & 0 \end{pmatrix}, \quad (19)$$

with  $G_{WT}^{-1} \equiv G_{\pi\rho}^{-1} - v_{WT}$ . We numerically find that  $G_{WT}$  has single resonance pole above the  $\pi\rho$  threshold as

$$G_{WT} = \frac{G_{\pi\rho}}{1 - v_{WT}G_{\pi\rho}} \equiv \frac{Z(s)}{s - s_p}. \quad (20)$$

This pole appears due to non-perturbative dynamics between  $\pi$  and  $\rho$  through the 4-point coupling  $v_{WT}$ , so that we interpret  $(s - s_p)^{-1}$  in Eq. (20) as the propagator of “ $\pi\rho$ -molecule state” with a wave function renormalization factor  $Z(s)$ . To renormalize (20),  $Z(s)$  can be attached to the interaction sector by  $\bar{G}^{-1} \equiv \text{diag}(\sqrt{Z}, 1)G^{-1}\text{diag}(\sqrt{Z}, 1)$  as

$$\bar{G}^{-1} = \begin{pmatrix} s - s_p & 0 \\ 0 & s - m_{a_1}^2 \end{pmatrix} - \begin{pmatrix} 0 & \sqrt{Z}v_{a_1\pi\rho} \\ \sqrt{Z}v_{a_1\pi\rho} & 0 \end{pmatrix}, \quad (21)$$

where the first term is the inverse of the “free” propagator for the  $\pi\rho$ -molecule state and the  $q\bar{q}$  state as the  $a_1$  meson. Now, by solving the relativistic eigenvalue equation for  $\bar{G}$  as

$$\det \bar{G}^{-1} = 0, \quad (22)$$

we have arrived at two-level model for the  $a_1$  meson with the  $\pi\rho$ -molecule and the  $q\bar{q}$  components having proper mixing.

Now, to get the geometrical map on the complex- $N_c$  plane for the  $a_1$  meson, we reduce Eq. (22) to the Schrödinger equation for Eq. (1), with a non-relativistic approximation. We approximate the molecule propagator and the renormalization factor in Eq. (20) as  $(s - s_p)^{-1} \simeq \{2\sqrt{s_p}(E - \sqrt{s_p})\}^{-1}$  and  $\sqrt{Z} \simeq 84 - 21i$  estimated at  $\sqrt{s} = \sqrt{s_p} \simeq 1012 - 221i$  in MeV unit. We also approximate the  $q\bar{q}$  propagator and the coupling constant as  $(s - m_{a_1}^2)^{-1} \simeq \{2m_{a_1}(E - m_{a_1})\}^{-1}$  and  $v_{a_1\pi\rho} \simeq -6493$  at  $\sqrt{s} = m_{a_1} = 1189$  in MeV unit. Such energy fixing has been traditionally employed, e.g., in nuclear-physics shell-model study, where the absorptive effects into decay channels outside of the model space are represented by the non-hermite matrix elements [15]. Then, Eq. (22) can be written as

$$(E - \sqrt{s_p})(E - m_{a_1}) - \frac{1}{(2\tilde{m})^2}(\sqrt{Z}v_{a_1\pi\rho})^2 = 0, \quad (23)$$

with  $\tilde{m} \equiv \sqrt{\sqrt{s_p}m_{a_1}}$ . From the Schrödinger equation (23), we can construct the two dimensional Hamilton matrix as

$$\mathcal{H} = \begin{pmatrix} \frac{1}{2\tilde{m}}\sqrt{\sqrt{s_p}} & \frac{1}{2\tilde{m}}\sqrt{Z}v_{a_1\pi\rho} \\ \frac{1}{2\tilde{m}}\sqrt{Z}v_{a_1\pi\rho} & m_{a_1} \end{pmatrix}. \quad (24)$$

Now we evaluate  $N_c$ -counting for the matrix elements in Eq. (24). According to large- $N_c$  QCD [8, 9],  $m_{a_1}$ ,  $v_{a_1\pi\rho}$  and  $G_{\pi\rho}$  have  $N_c$ -dependence as

$$m_{a_1} \sim O(N_c^0), \quad v_{a_1\pi\rho} \sim O(N_c^{-1/2}), \quad G_{\pi\rho} \sim O(N_c^0). \quad (25)$$

For energy region far from the threshold;  $s \gg (m_\rho + m_\pi)^2$ , the Weinberg-Tomozawa interaction (15) can be simplified as  $v_{WT} \sim s \times O(N_c^{-1})$  as the mesonic four-point interaction [8, 9]. Therefore, Eq. (20) can be rewritten as

$$G_{WT} \sim \frac{G_{\pi\rho}}{1 - \{s \times O(N_c^{-1})\}G_{\pi\rho}} \sim \frac{O(N_c)}{s - O(N_c)/G_{\pi\rho}}. \quad (26)$$

By comparing Eqs.(20) and (26), we can also estimate the  $N_c$  dependence of  $\sqrt{s_p}$  and  $\sqrt{Z}$  as

$$\sqrt{s_p} \sim O(N_c^{1/2}), \quad \sqrt{Z} \sim O(N_c^{1/2}), \quad (27)$$

where energy dependence of the loop function  $G_{\pi\rho}$  is approximately ignored. By using Eq. (27), we can also estimate the  $N_c$ -dependence of energy scale  $\tilde{m}$  introduced in Eq. (23) as

$$\tilde{m} \sim O(N_c^{1/4}). \quad (28)$$

By using the Eqs. (25), (27) and (28) for the matrix elements in Eq. (24), we eventually get the Complex 2D Matrix Model



for  $a_1$  meson with  $N_c$  dependence factored out by  $\lambda$  as

$$\mathcal{H}(\lambda) = \begin{pmatrix} \frac{1}{\lambda^2} \sqrt{s_p} & \frac{\lambda}{2\tilde{m}} \sqrt{Z} v_{a_1 \pi \rho} \\ \frac{\lambda}{2\tilde{m}} \sqrt{Z} v_{a_1 \pi \rho} & m_{a_1} \end{pmatrix}, \quad (29)$$

$$\lambda \equiv \left( \frac{3}{N_c} \right)^{1/4}, \quad (30)$$

where  $\sqrt{s_p}$ ,  $\sqrt{Z}$ ,  $v_{a_1 \pi \rho}$  and  $\tilde{m}$  in (29) are the constants estimated at  $N_c = 3$  as shown above Eq. (23).  $\frac{1}{\lambda^2} \sqrt{s_p}$  and  $m_{a_1}$  in Eq. (29) are the energies of the  $\pi\rho$ -molecule state and the  $q\bar{q}$  state as the  $a_1$  meson, and they form the appropriate basis. The (1,1) element with negative power of  $\lambda$  reflects that a resonance state appears due to highly nonperturbative hadron dynamics. In fact, the building block (29) with the  $\lambda$  dependence is *universal* for any hadron with admixed natures of the mesonic molecule and the  $q\bar{q}$  components.

Then, by applying the conditions (9) and (11) to the Hamiltonian (29), we can get the geometrical map on the complex- $N_c$  plane for the  $a_1$  meson in Fig. 4. Six exceptional points and four transition lines, two of which are half-lines, appear on this map. These numbers can be derived from the power counting about  $\lambda$  in Eq. (12). The transition line shown by the solid curve can cross the real  $\lambda$  axis between  $\lambda = 0$  ( $N_c = \infty$ ) and  $\lambda = 1$  ( $N_c = 3$ ). The crossing point shows a critical color number for transition as  $\lambda_t = (3/N_c)^{1/4} \sim 0.93$ , i.e.,  $N_c \sim 4.0$ . This result indicates that, with continuous change of  $N_c$  from  $\infty$  to 3, the internal structures of two hadronic eigenstates can be exchanged in terms of appropriate basis  $q\bar{q}$  and  $\pi\rho$ -molecule at the critical color number  $N_c \sim 4.0$ . Such a critical color number with character exchange for the  $a_1$  meson is also reported from the analysis of the pole residues in Ref. [22]. In this way, by looking into the existence of nature transition from the geometry on the complex- $N_c$  plane, we can successfully know the internal structure of hadrons with variation of  $N_c$  from  $\infty$  to 3.

In summary, we have formulated the Complex 2D Matrix Model to discuss the nature transition between two resonance states. We suggest that the geometry on the complex-parameter space will give a simple criterion of nature transition between resonance states within the real-parameter subspace. By applying the model to hadron physics, we have calculated the critical color number of the nature transition between hadronic states, which can be used to know the internal structure of hadrons with variation of  $N_c$  from  $\infty$  to 3. We show that, due to the development of hadron interactions scaled by  $1/N_c$ ,  $q\bar{q}$  state can become dominated by mesonic-molecule component, while mesonic molecule state can become dominated by  $q\bar{q}$  component around  $N_c = 3$ . We hope that the new concept of geometry on the complex- $N_c$  plane will shed light on the exotic physics in QCD for the future.

Our model can be employed to general multi-level problems of resonances to analyze their internal structures with variation of a parameter in each system. Wide applications of our model to resonance physics are expected as a future prospect.

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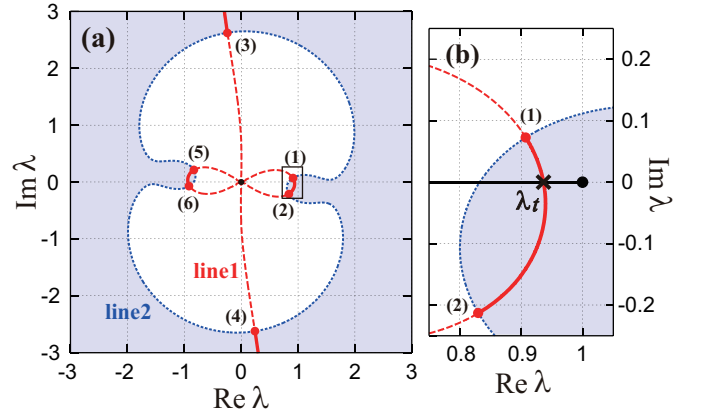


FIG. 4: (color online). (a) Geometrical map on the complex- $N_c$  plane with  $\lambda = (3/N_c)^{1/4}$ . Constants in Eq. (29) are  $\sqrt{s_p} = 1012 - 221i$ ,  $m_{a_1} = 1189$ ,  $\sqrt{Z} = 84 - 21i$  and  $v_{a_1 \pi \rho} = -6493$  in MeV unit. Line 1 and shaded area with the boundary of line 2 correspond to the conditions (9) and (11), respectively. Six exceptional points ( $n$ ) ( $n = 1 \sim 6$ ) as the crossing points between line 1 and line 2, and four transition lines as solid curves appear. (b) Close-up figure around a blank square in (a). Transition line as a solid curve crosses the real axis at  $\lambda_t \sim 0.93$ , i.e.,  $N_c \sim 4.0$ , which locates between  $\lambda = 0$  ( $N_c = \infty$ ) and  $\lambda = 1$  ( $N_c = 3$ ).

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